Graph Games on Ordinals*

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ABSTRACT.
We consider an extension of Church's synthesis problem to ordinals by adding limit transitions to graph games. We consider game arenas where these limit transitions are defined using the sets of cofinally visited states. In a previous paper, we have shown that such games of ordinal length are determined and that the winner problem is PSPACE-complete, for arenas where the length of plays is always smaller than ω^ω. However, the proof uses a rather involved reduction to classical Muller games, and the resulting strategies need infinite memory.
We adapt the LAR reduction to prove the determinacy and to generate strategies with finite memory in the general case, using a reduction to games where the limit transitions are defined by priorities. We provide an algorithm for computing the winning regions of both players in these games, with a complexity similar to parity games. Its analysis yields three results: determinacy without hypothesis on the length of the plays, existence of memoryless strategies, and membership of the winner problem in NP $\cap$ co-NP.

1 Introduction

Church's problem, introduced in [Chu63], is fundamental in the theory of automata over infinite strings. It considers a specification $\phi(X, Y)$ — usually a MSO formula — over pairs of infinite sequences. A solution to this problem is a circuit which computes an output sequence Y using a letter-by-letter transformation of the input sequence X. The Büchi-Landweber theorem shows the decidability of this problem, and provides an automatic procedure to compute a solution [BL69]. The proof builds on McNaughton's game-theoretic presentation of this problem. McNaughton games are perfect information two-player games where at every stage $n < \omega$, player X chooses first whether he accepts n, and Y replies in kind. Player Y wins a play if the sets of integers accepted by X and Y verify ϕ. A winning strategy for Y gives a solution to Church's problem. Additionally, a winning strategy can be computed by a finite ω-automaton with output or, equivalently, defined using an MSO formula.

Church's problem can be extended to sequences of arbitrary ordinal length. One possible extension is to fix the length of the plays in advance; Rabinovich and Shomrat show in [RS08], using a composition method, that for any countable ordinal α and MSO formula $\phi(X, Y)$, the corresponding McNaughton game is determined and the winner problem is decidable. Moreover, if $\alpha < \omega^\omega$ it's possible to compute a formula defining a winning strategy.

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Our approach is based on the graph games used in verification [Tho95], and Büchi's automata on words of ordinal length [Bü73]. Limit transitions allow the game to continue past any limit ordinal. In this model, the length of the plays is not fixed a priori, but depends on the actions of the players: the game stops when one of the players wins. In the paper, we only consider reachability winning conditions; however, any regular condition can be represented this way, because the addition of limit transitions allows to embed more complex transitions in the game arena itself. In [CH08], we studied a restriction of these games, disallowing limit transitions of the form $P \to q \in P$, which reduces the scope of this work to plays shorter than ω^ω. Another drawback is that the strategies we obtained needed infinite memory. In this paper, we lift this restriction and prove the determinacy and existence of strategies with finite memory. We first solve a particular case of games where the limit transitions are defined using priorities, which are closely related to parity games of length ω. We present an algorithm computing the winning regions for both players in these games. Determinacy follows from its correctness, and further analysis gives positional winning strategies for both players in their winning regions. We derive from this the membership of the winner problem in $\mathsf{NP} \cap \text{co-}\mathsf{NP}$. We also derive an alternative proof of the results of Rabinovich and Shomrat on McNaughton games of length smaller than ω^ω: determinacy, decidability, definable strategy and strategy synthesis. Using an adaptation of the *Latest Appearance Records* of Gurevich and Harrington [GH82], we give a reduction from ordinal games to priority ordinal games. From this extended LAR reduction and our former results, we derive the determinacy of games of ordinal length, as well as the existence of finite-memory strategies for both players.

Overview of the paper. In Section 2, we recall the definitions of graph games of ordinal length, as well as their variant with priority-controlled transitions. Section 3 presents an algorithm for solving priority ordinal games, and its theoretical consequences. Section 4 shows how to adapt the LAR reduction to games of ordinal length, in order to get general determinacy and finite-memory strategies for all ordinal games. Finally, Section 5 summarises our results, and presents perspectives for future work.

2 Definitions

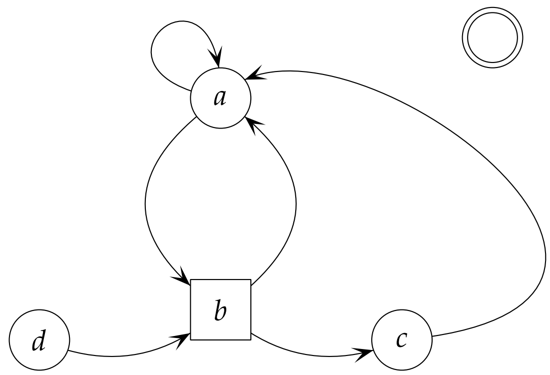
Ordinals and words of ordinal length

An ordinal α is a set equipped with a well-founded linear order. Ordinals can be ordered in a natural way, and one can consider each ordinal as the set of the smaller ordinals. A *word* ρ of length α over an alphabet Σ is a mapping $(\rho_\beta)_{\beta<\alpha}$ from α to Σ. We denote by $|\rho|$ the length of the word. The *prefix* of ρ of length $\beta \leq |\rho|$, noted $\rho_{<\beta}$, is defined as $(\rho_\gamma)_{\gamma<\beta}$. Similarly, we write $\rho_{\geq\beta}$ for the *suffix* $(\rho_\gamma)_{\beta\leq\gamma<|\rho|}$. The subset of Σ appearing in ρ is $\mathrm{Occ}(\rho) = \{\rho_\beta \mid \beta < |\rho|\}$. Finally, if $|\rho|$ is a limit ordinal, the *limit* of ρ, noted $\lim \rho$, is the set $\{s \in \Sigma \mid \forall\beta < |\rho|,\ \exists\gamma > \beta,\ \rho_\gamma = s\}$.

Games of ordinal length

A reachability game of ordinal length (*ordinal game*) G is played by two players called Eve and Adam on an *arena* of the form $(Q, Q_E, Q_A, T, \lambda)$. The tuple (Q, T) is a directed graph and the set of vertices is partitioned between Adam's vertices (Q_A, represented by

□), Eve's vertices (Q_E, represented by ○), and a *target* vertex ◎. The function λ represents *limit transitions*; it maps $\mathcal{P}(Q_A \cup Q_E)$ to Q. We assume that every vertex except ◎ has at least one successor. We give an example of reachability game of ordinal length in Figure 1. In this paper, we consider the case where Q is finite.



$$\lambda(\{a\}) = d \qquad \lambda(\{a,b\}) = c \qquad \lambda(\{a,b,c\}) = c$$
$$\lambda(\{a,b,d\}) = a \qquad \lambda(\{a,b,c,d\}) = \circledcirc$$

Other limit transitions aren't possible in this example

Figure 1: Game of ordinal length

A *play* ρ on a game G is an ordinal word on Q such that for any $\alpha < |\rho|$:

- if $\alpha = \beta + 1$, then $(\rho_\beta, \rho_\alpha) \in T$;
- if α is a limit ordinal, then $\lambda(\lim \rho_{<\alpha}) = \rho_\alpha$.

The set of all plays is noted Ω. It can be divided into four disjoint subsets:

1. The set of plays which have a last state in Q_E. These plays can be extended through a successor transition, chosen by Eve — a *move* of Eve.
2. The set of plays which have a last state in Q_A. These plays can be extended through a successor transition, chosen by Adam — a *move* of Adam.
3. The set of plays which have ◎ as last state. These plays are said to be winning for Eve. Any other play is winning for Adam.
4. The set of plays without a last state. These plays can be extended through a unique limit transition.

Notice that our definition for winning plays deviates from the classical interpretation of infinite games, where there is no winner for the partial plays. This is necessary here, as non-winning plays can go on without ever ending, even in the transfinite sense, so it's not possible to easily distinguish between plays where Eve has not yet won, and plays that are won by Adam.

A *strategy* for Eve is a function $\sigma : \Omega \to Q$ such that if ρ ends in $q \in Q_E$, then $(q, \sigma(\rho)) \in T$. A strategy with finite memory M for Eve is a finite transducer working over the states of M. It is defined by three functions:

- $\mu : M \times Q \to M$ is the memory update for successor transitions.
- $\theta : \mathcal{P}(M) \to M$ is the memory update for limit transitions.

- $\nu : M \times Q_E \to Q$ outputs Eve's next move.

One can define in the same way strategies and strategies with finite memory for Adam. If M has only one element, σ is a *positional* strategy. A play ρ of length α is *consistent* with a strategy σ for Eve if $\rho_{\beta+1} = \sigma(\rho_{<\beta})$ for every β such that $\beta + 1 < \alpha$ and $\rho_\beta \in Q_E$. A strategy σ is *winning* for Eve if there is an ordinal α such that any play consistent with σ has length less than α. Notice that this condition imposes that $\circledcirc$ is eventually reached, as plays can otherwise always be extended. Conversely, a strategy τ is winning for Adam if no play consistent with τ ends in $\circledcirc$. A game is *determined* if there is always a winning strategy for one of the players.

If all limit transitions lead either to $\circledcirc$ or to a sink state, then all plays are shorter than ω, and the game is a traditional Muller game.

In [CH08], we showed that a subclass of ordinal games are determined:

THEOREM 1.*[[CH08]] Reachability games of ordinal length without transitions of the form $\lambda(P) = q \in P$ are determined.*

Notice that this subclass is not a mere syntactic condition: it restricts the scope of Theorem 1 to games where the plays have length less than ω^ω, as can be deduced from Theorem 2:

THEOREM 2.*[[Cho78]] In an automaton with n states where no limit transition is of the form $\lambda(P) = q \in P$, all runs are shorter than ω^n.*

Priority ordinal games. A reachability game of ordinal length with priority transitions (*priority ordinal game*) is a game where the limit transition function λ is defined in a specific way: a *colouring function* χ maps each state to a colour in $\{0, \dots, d-1\}$; another function δ maps each colour to a state of Q. Then, for any set $P \subseteq Q$, the limit transition is given by $\lambda(P) = \delta(\min\{\chi(q) \mid q \in P\})$.

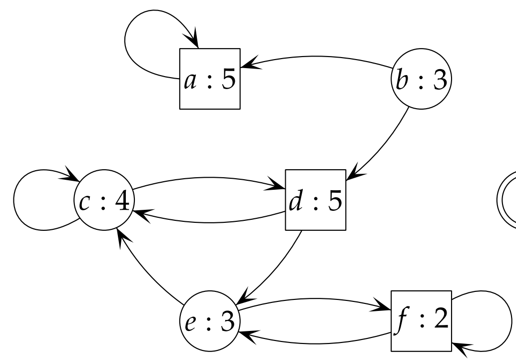
Figure 2 gives an example of a priority game, with $d = 6$. In this game, Adam wins from states c and d: from d he can go to c, and any limit transition will take the token back to d. Eve wins from everywhere else: from b she goes to a and the token will reach the target after playing $(ba^\omega)^\omega$; from e she goes to f and the token will eventually reach a.

Remarks: Not all ordinal games can be represented by priority transitions. In the game of Figure 1, for example, the set $\{a, b, c, d\}$ leads neither to the destination of $\{a, b, c\}$, nor to the one of $\{a, b, d\}$. This cannot occur in a priority game: the destination $\min \chi(\{a, b, c, d\})$ would be either $\min \chi(\{a, b, c\})$, or $\min \chi(\{a, b, d\})$.

Subgames, attractors and traps. We recall here some classical concepts for infinite games (see e.g. [Tho95]), which we use in the context of ordinal games.

Let $G = (Q, Q_E, Q_A, T, \lambda)$ be an ordinal game, and Q' a subset of Q. The tuple $G' = (Q', Q'_E, Q'_A, T', \lambda')$ —where $Q'_E = Q' \cap Q_E$, $Q'_A = Q' \cap Q_A$, and T' and λ' are the restrictions to Q' of the transitions in G— is an *ω-subgame* of G if every state in G' has a successor. We write $G' = G \setminus P$ if $Q' = Q \setminus P$.

Let P be a subset of states in a game G. The *ω-attractor* to P for Eve, noted $\mathrm{Attr}_E^G(P)$, is the set of states such that Eve can ensure that P is reached after a finite number of moves; it is defined as $\bigcup_{i \geq 0} \mathrm{Attr}_i$, where:



$\delta(2) = a \qquad \delta(3) = \circledcirc \qquad \delta(4) = d \qquad \delta(5) = b$

Figure 2: Priority Game

- $\mathrm{Attr}_0 = P$;
- $\mathrm{Attr}_i \subseteq \mathrm{Attr}_{i+1}$;
- if $q \in Q_E$ and q has a successor in Attr_i, then $q \in \mathrm{Attr}_{i+1}$;
- if $q \in Q_A$ and all successors of q in G are in Attr_i, then $q \in \mathrm{Attr}_{i+1}$.

An ω*-trap* for Eve is a subset P of states such that Adam can ensure that the token stays in P for at least ω moves:

- if $q \in P \cap Q_E$, all its successors are in P;
- if $q \in P \cap Q_A$, q has a successor in P.

PROPOSITION 3. *In an ω-attractor for Eve, she has a positional strategy to ensure that P is reached in a finite number of moves.*

Remarks: In all this paper, we use the terms *attractor* and *trap* to refer to ω-attractors and ω-traps, without any assumption about what happens beyond ω. Computing ordinal attractors is, indeed, the point of this work.

3 Solving priority ordinal games

Algorithm. Our main result is an algorithm computing the winning regions of both players in a priority ordinal game. It is inspired by Zielonka's algorithm for infinite (ω-length) parity games in [Zie98].

In order to make the algorithm simpler, we assume without loss of generality that the target $\circledcirc$ is the only state with priority 0, and that no state has priority 1. The value of $\delta(1)$ is thus unused, and we assume that it is not defined.

The algorithm uses two arrays, $\mathtt{v}$ and To, indexed by colours. For each colour i, $\mathtt{v}[i] \in \{A, E\}$ represents the player "controlling" i at a given point in the algorithm. $\mathrm{To}[i] \subseteq Q$ is a set of states where player $\mathtt{v}[i]$ can guarantee an invariant which will be precised later. Figure 3 presents this algorithm in pseudo-code.

We compute embedded attractors, starting with the smallest (*i.e.* most important) colour: we compute an attractor to that colour, then remove these states from the graph

Input: The game G
Output: The winning regions of Eve and Adam

```
1  v[0] ← E
2  To[0] ← Attr_E^G(χ^{-1}(0))
3  for 0 < i < d do To[i] ← ∅
4  H ← G \ To[0]
5  i ← 0
6  while (To[0] ∪ To[1]) ≠ G do
7      while (H ≠ ∅) do
8          i++
9          if ∃j | δ(i) ∈ To[j] then
10             v[i] ← v[j]
11         else
12             v[i] ← A
13         To[i] ← Attr_{v[i]}^H(χ^{-1}(i))
14         H ← H \ To[i]
15     Tmpto ← To[i]
16     Tmpv ← v[i]
17     repeat
18         H ← H ∪ To[i]
19         To[i] ← ∅
20         v[i] ← A
21         i--
22     until (v[i] = Tmpv)
23     To[i] ← To[i] ∪ Attr_{Tmpv}^H(Tmpto)
24     H ← H \ To[i]
25 return (To[0], To[1])
```

Figure 3: Algorithm for Priority Games

and start again with the next colour. When all the graph is covered, the last computed attractor, $\mathrm{To}[k]$, is merged with a former one, $\mathrm{To}[j]$. We recompute then the attractors to colours greater than j. The algorithm ends when all the states are either in $\mathrm{To}[0]$ or in $\mathrm{To}[1]$. The former contains the winning region for Eve, and the latter is the winning region for Adam. The termination is guaranteed by the fact that a state can only be removed from "$\mathrm{To}[i]$" when another is added to "$\mathrm{To}[j]$" with $j < i$, and that whenever line 14 is reached, every state belongs to exactly one of the $\mathrm{To}[i]$.

The major difference with Zielonka's algorithm is that we have to determine which player wants to reach j. This information is stored in $\mathtt{v}[j]$, and can change each time the attractor to j is computed: $\mathtt{v}[j] = E$ if and only if there is a smaller colour j' such that $\mathtt{v}[j'] = E$ and $\delta(j) \in \mathrm{To}[j']$.

Correctness. In order to prove the correctness of the algorithm, we use the notation H^j for

$Q \setminus \cup_{k<j} \mathrm{To}[k]$, and the following easy but useful properties obtained from the construction of the array To.

PROPOSITION 4. *H^j is an ω-subgame of G.*

PROPOSITION 5. *Let j be a colour, then* $\mathrm{Attr}^{H^j}_{\mathtt{v}[j]}(\mathrm{To}[j]) = \mathrm{To}[j]$.

The correctness of the algorithm is proved separately for the two players: the arguments involved, while close, cannot be easily unified. In both cases, however, we define a predicate referring to the plays of the game and a property derived from it, and show that the property holds along the whole run. Its interpretation at the end of a run implies correctness.

In Eve's proof, we use the loop invariant $\mathcal{I}$ and the predicates $\mathcal{M}^j$ on the plays. The corresponding predicates for Adam are $\mathcal{J}$ and $\mathcal{N}^j$, respectively.

Informally, the predicates $\mathcal{M}^j$ correspond to strong **until** predicates of the form "$(> j)\mathcal{U}(j)$". Adam's predicates $\mathcal{N}^j$ correspond to weak **until** predicates of the form "$(> j)\mathcal{W}(j)$".

DEFINITION 6. *The predicate $\mathcal{M}^j(\rho)$ is defined as "$\mathtt{v}[j] = E$, and $\exists \alpha < \omega^{d-j}$ such that* $\mathrm{Occ}(\rho_{<\alpha}) \subseteq \mathrm{To}[j]$ *and either:*

- *$|\rho| = \alpha$, or*
- *$\rho_\alpha \in \mathrm{To}[j] \cap \chi^{-1}(j)$, or*
- *$\exists k < j$ such that $\mathcal{M}^k(\rho_{\geq\alpha})$ holds".*

DEFINITION 7. *$\mathcal{I}$ is the property "Eve has a positional strategy σ such that, for any j such that $\mathtt{v}[j] = E$, and any play ρ starting in* $\mathrm{To}[j]$ *and consistent with σ, $\mathcal{M}^j(\rho)$ holds".*

$\mathcal{I}$ holds at the beginning of a run: before line 6, To[0] is the only non-empty set, and it contains only $\mathrm{Attr}_E(\chi^{-1}(0))$. Propositions 8 and 9 guarantee that it remains true throughout the execution.

PROPOSITION 8. *Let us suppose that $\mathcal{I}$ holds at line 8. Then $\mathcal{I}$ holds at the next visit of line 14.*

SKETCH OF PROOF. If $\mathtt{v}[i] = A$, the property $\mathcal{I}$ is unchanged after the iteration. If $\mathtt{v}[i] = E$, an attractor strategy for Eve guarantees a visit to i in less than ω moves, unless Adam chooses to send the token outside of $\mathrm{To}[i]$. If he does that, the structure of the array To as a series of attractors guarantees that the token is sent to a $\mathrm{To}[j]$ such that $j < i$ and $\mathtt{v}[j] = E$. ∎

PROPOSITION 9. *Let us suppose that $\mathcal{I}$ holds before a visit to line 17. Then $\mathcal{I}$ holds at the next visit of line 24.*

SKETCH OF PROOF. Once again, the interesting cases are those where $Tmpv = E$ and the token remains in $Tmpto$ long enough. As $Tmpto$ is a trap for Adam, Eve's strategy guarantees an infinite number of visits to i (the corresponding colour) in less than ω^{d-i+1} moves. The token is then sent to $\delta(i)$ which by definitions of $\mathtt{v}$ and i belongs to a $\mathrm{To}[j]$ such that $j < i$ and $\mathtt{v}[j] = E$. ∎

The structure of the proof for the states of Adam is quite similar, although the predicates are slightly weaker. We can prove that $\mathcal{J}$ holds along the whole run.

Definition 10. *The predicate* $\mathcal{N}^j(\rho)$ *is defined as "*$\mathrm{v}[j] = A$ *and* $\exists\alpha$ *such that* $\mathrm{Occ}(\rho_{<\alpha}) \subseteq \mathrm{To}[j]$ *and either:*

- $|\rho| = \alpha$*, or*
- $\rho_\alpha \in \mathrm{To}[j] \cap \chi^{-1}(j)$*, or*
- $\exists k < j$ *such that* $\mathcal{N}^k(\rho_{\geq\alpha})$ *holds."*

Definition 11. $\mathcal{J}$ *is the property "Adam has a positional strategy* τ *such that, for any* j *such that* $\mathrm{v}[j] = A$*, and any play* ρ *starting in* $\mathrm{To}[j]$ *and consistent with* τ*,* $\mathcal{N}^j(\rho)$ *holds".*

Consequences. The interpretation of $\mathcal{I}$ and $\mathcal{J}$ at the end of a run leads to the following Theorem:

Theorem 12. *Let* $G = (Q, Q_E, Q_A, T, \chi, \delta)$ *be a priority ordinal game such that* $\chi(Q \setminus \circledcirc) \subseteq [2, d-1]$*. Then*

1. *G is determined;*
2. *Eve and Adam have positional winning strategies;*
3. *If Eve can win, then she can reach* $\circledcirc$ *in less than* ω^d *moves.*

Corollary 13. *The problem of the winner in reachability games of ordinal length with priority transitions belongs to* NP $\cap$ *co-*NP*.*

Proof. Let's consider a game $G = (Q, Q_E, Q_A, T, \chi, \delta)$ and a state $q \in Q$. The problem is "Does Eve have a winning strategy if the token starts in q?"
*co-*NP*-membership.* If q is not winning for Eve, a winning strategy for Adam is a polynomial counter-example. Using it, we define the automaton $\mathcal{A}_\tau$ from G by removing the successor transitions of the form (r, s) with $r \in Q_A$ and $s \neq \tau(r)$. If τ is winning, then $\mathcal{L}(\mathcal{A}_\tau) = \emptyset$. On the other hand, if q is winning for Eve, $\mathcal{L}(\mathcal{A}_\tau) \neq \emptyset$ for any τ. As the emptiness problem for ordinal automata is decidable in polynomial time [Col07], this is a co-NP procedure.
NP*-membership.* We guess a positional strategy σ for Eve, and use it to define the automaton $\mathcal{A}_\sigma$ by removing from G successor transitions of the form (r, s) with $r \in Q_E$ and $s \neq \sigma(r)$, and making every state except $\circledcirc$ final. The strategy σ is winning if and only if the language accepted by the product of $\mathcal{A}_\sigma$ and an automaton accepting runs of length greater than ω^d is empty. ∎

This algorithm also yields an alternate proof of the following theorem by Rabinovich and Shomrat, in the case of ordinals less than ω^ω. They consider two-player games $\mathcal{G}^\alpha_\phi$ defined by an MSO formula ϕ and an ordinal α. In such a game, each player builds a subset of α in the following way: for every ordinal $\beta < \alpha$, player 0 chooses whether he wants to pick β, and player 1 responds. The set of positions picked by player 0 is noted X, those positions chosen by player 1 form the set Y. Player 0 wins if $\phi(X, Y)$ is true.

Theorem 14.*[Theorem 29 of [RS08]] Let* α *be a countable ordinal and* $\phi(X, Y)$ *be a MSO-formula.*

Determinacy: One of the players has a winning strategy in the game $\mathcal{G}^\alpha_\phi$*.*
Decidability: It is decidable which of the players has a winning strategy.
Definable strategy: If $\alpha < \omega^\omega$*, then the player who has a winning strategy also has a definable winning strategy. For every* $\alpha \geq \omega^\omega$*, there is a formula for which this fails.*

Synthesis algorithm: If $\alpha < \omega^\omega$, we can compute a formula $\psi(X, Y)$ that defines a winning strategy for the winning player in $\mathcal{G}^\alpha_\phi$.

PROOF. Both the MSO-formula ϕ and the ordinal $\alpha < \omega^\omega$ can be represented as finite automata over ordinal words with priority transitions. The automaton corresponding to ϕ gets pairs (X_i, Y_i) of letters as input. The automaton for α accepts words of length exactly α. The product of these two automata can be seen as an ordinal game, where at every step $i < \alpha$ Adam chooses X_i and Eve then chooses Y_i. Eve needs to ensure that after α moves from each player, the ϕ automaton is in an accepting state, and thus she has a winning strategy if and only if Y has a winning strategy in $\mathcal{G}^\alpha_\phi$.

Determinacy and decidability follow from the correctness of Algorithm 3. Positional strategies in this finite arena can be represented as finite automata, and thus are definable. A synthesis algorithm can be derived from the proof of Algorithm 3. ∎

Notice that it is not possible to represent a single ordinal greater than ω^ω as a finite automaton on ordinal words. Thus, we can't interpret McNaughton games of length greater than ω^ω, as defined in [RS08], as finite graph games, and our approach doesn't provide an easy proof of the above theorem for this case.

4 From priority transitions to ordinal games

In this section, we extend the results of Section 3 to the more general case of games of ordinal length, through an *ordinal game reduction*: there is a bisimulation between the graphs, such that equivalent states belong to the same player, and equivalent plays have limit transitions to equivalent states.

The LAR reduction. The Latest Appearance Records (LAR) were introduced by Gurevich and Harrington [GH82], in order to prove the *Forgetful Determinacy* of Muller ω-games (*i.e.* the existence of finite memory strategies). A LAR for a game G with n states is a pair (π, i), where π is a permutation over the states of G and i is an integer such that $1 \leq i \leq n$. We reduce any ordinal game $G = (Q, Q_E, Q_A, T, \lambda)$, to a priority ordinal game $\mathbb{G} = (\mathbb{Q}, \mathbb{Q}_E, \mathbb{Q}_A, \mathbb{T}, \chi, \delta)$. The states and successor transitions are defined in the same way as in the original reduction:

- $\mathbb{Q} = \{(\pi, i) \mid \pi \text{ is a permutation over } Q, 1 \leq i \leq n\}$
- $\mathbb{Q}_E = \{(\pi, i) \in \mathbb{Q} \mid \pi(1) \in Q_E\}$
- $\mathbb{Q}_A = \{(\pi, i) \in \mathbb{Q} \mid \pi(1) \in Q_A\}$
- $(\pi, i) \xrightarrow{\mathbb{T}} (\pi', i')$ if and only if:
  - $\pi(1) \xrightarrow{T} \pi'(1)$
  - $\forall q, r \in Q \setminus \{\pi'(1)\},\ \pi^{-1}(q) < \pi^{-1}(r) \Leftrightarrow \pi'^{-1}(q) < \pi'^{-1}(r)$
  - $\pi(i') = \pi'(1)$

The limit transitions, by contrast, are much more involved than in the infinite setting. As we need to keep track of some of the memory *after* a limit transition, we use a different colour for each state of $\mathbb{G}$ — ordered so that $i < i' \Rightarrow \chi(\pi, i) > \chi(\pi', i')$ (the exact ordering is unimportant, as long as this condition is verified). By abuse of notation we describe δ as a function from $\mathbb{Q}$ to $\mathbb{Q}$: $\delta(\pi, i)$ is the LAR (π', i') such that:

- $\lambda(\cup_{j=1}^{i}\{\pi(j)\}) = \pi'(1)$
- $\forall q, r \in Q \setminus \{\pi'(1)\},\ \pi^{-1}(q) < \pi^{-1}(r) \Leftrightarrow \pi'^{-1}(q) < \pi'^{-1}(r)$
- $\pi(i') = \pi'(1)$

The target states for Eve are the states (π, i) such that $\pi(1)$ is the target state in the original game G. Obviously these states can be merged to get a unique target state.

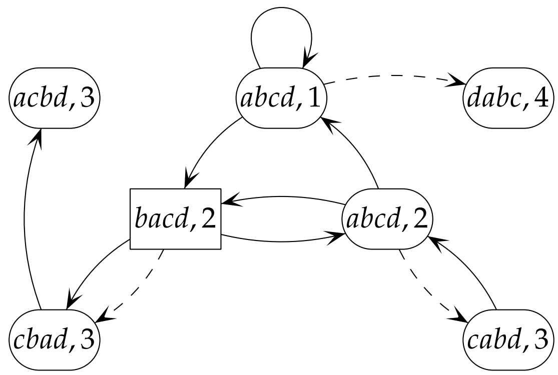


Figure 4: Detail of the LAR reduction of the game of Figure 1

Figure 4 gives a detail of the reduction of the game of Figure 1 (dashed arrows represent limit transitions). The game bisimulation is proved as in the case of infinite games, with some added fun due to the limit transitions.

LEMMA 15. *There is a bisimulation between G and $\mathbb{G}$ such that two bisimilar plays without a last state have a limit transition to two bisimilar states.*

Results for ordinal games. The LAR reduction allows us to extend Theorem 12 to any ordinal game:

THEOREM 16. *Let $G = (Q, Q_E, Q_A, T, \lambda)$ be an ordinal reachability game with n states. Then*

1. *G is determined;*
2. *Eve and Adam have winning strategies with memory $n!$;*
3. *if Eve can win, then she can reach $\circledcirc$ in less than $\omega^{n!}$ moves.*

SKETCH OF PROOF. Theorem 16 follows from Theorem 12 and the LAR reduction. The LAR reduction preserves the winner, which gives us the determinacy. The third part is ensured because the length of plays is preserved. For the second part, we translate a positional strategy σ in $\mathbb{G}$ into a strategy $\varsigma = (\mu, \theta, \nu)$ in G:

- the memory states are LARs: $M = \mathbb{Q}$
- the memory updates mimic the transitions of $\mathbb{G}$:
  - $\mu((\pi, i), q) = \{(\pi', i') \in \mathbb{T}(\pi, i) \mid \pi'(1) = q\}$
  - $\theta(P) = \delta(\min\{\chi(P)\})$
- the next-move function follows σ: $\nu((\pi, i), q) = \sigma(\mu((\pi, i), q))$

This amounts to considering an equivalent play on the reduced game $\mathbb{G}$, keeping the whole LAR in memory. ∎

5 Conclusion

We present a new model of games with plays of ordinal length. Our model is not comparable in general with McNaughton games: on one hand, we do not impose a length a priori, but on the other hand we cannot be precise beyond ω^ω. However, in the case of games of length less than ω^ω, these two models are close, as shows our alternative proof of Rabinovich and Shomrat's theorem from [RS08].

In comparison to [CH08], we remove the hypothesis on the transitions, which limited the scope of our games to plays of length less than ω^ω. We also consider the central problem of arenas with priority transitions, and derive from their study some interesting results, for example the existence of strategies with finite or no memory. This leads to our solution to Church's synthesis problem; and the existence of a bound on the number of steps needed to reach the target.

One of our objectives is to use this formalism in the context of verification of timed open systems. This would allow us to deal with Zeno behaviours, whereas most of the time, the results about the subject forbid them [dAFH+03], or consider only non-Zeno runs [AM99]. We would like to have a more constructive approach on the problem, following for example [JT07].

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