Reachability Games of Ordinal Length

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LIAFA - Paris

January 21st, 2008

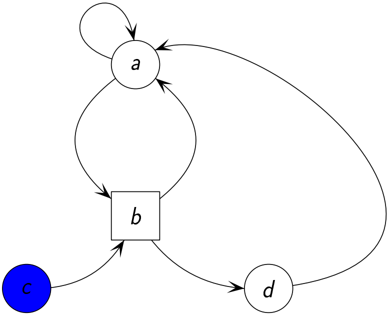
Plan

Motivation

- Games: useful for verification, controller synthesis
  - One player (Eve) corresponds to the system, the opponent (Adam) represents the system
- Zeno behaviours: timed systems, real time models, . . .
- Modelisation of systems where an unbounded number of events happen in finite time

Games (1)

- Finite graph $G = (V, E)$
- Partition $V = V_E \cup V_A$
- 2 players, Adam and Eve; Adam plays in V_A and Eve in V_E
- Winning condition



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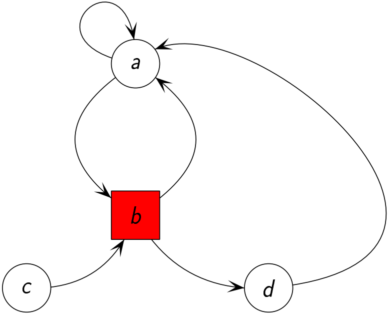

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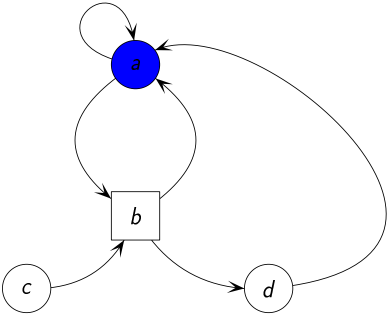

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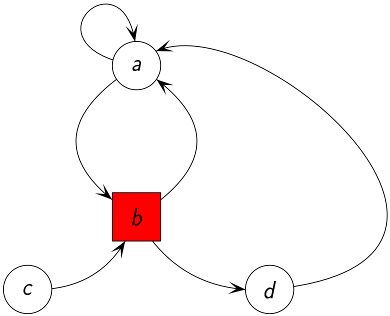

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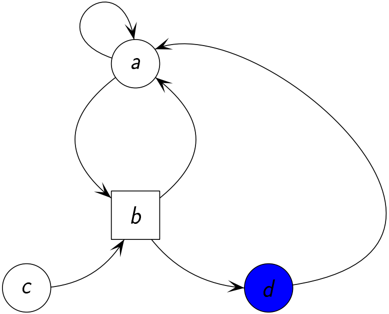

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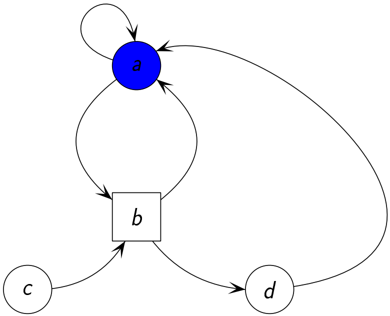
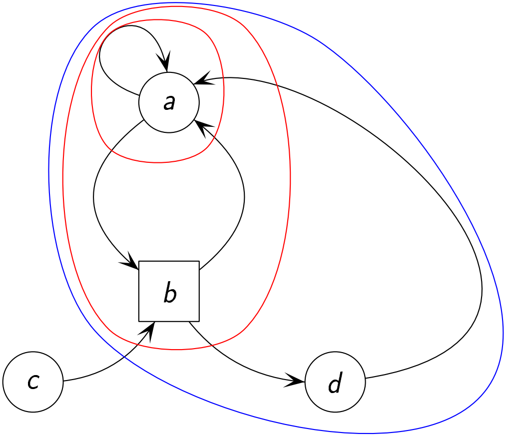


Muller games

Winning condition: Eve wins if the set of states visited infinitely often is in $\mathcal{F}$.

Example: Eve wins $\{a, b, d\}$, Adam wins $\{a, b\}$ and $\{a\}$.



A play is an infinite word, like *cbabdababdabdababababda* . . . .

Problems

A game is given by a partitioned graph and a winning condition.
We want to know:

- whether the game is determined (one of the players has a winning strategy);
- given an initial state, which is the winning player;
- how to compute a winning strategy.

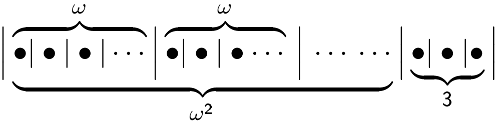
Muller games are determined (Martin), finding the winner is PSPACE-complete (Hunter and Dawar).

Beyond ω

We want models of systems where infinitely many actions can happen in finite time.

Examples:

- ω
- $\omega^2 + 3$



A play is now a word of ordinal length, such as $((ab)^\omega c)^\omega (ba)^\omega d$

Games

We add limit transitions to the arena.

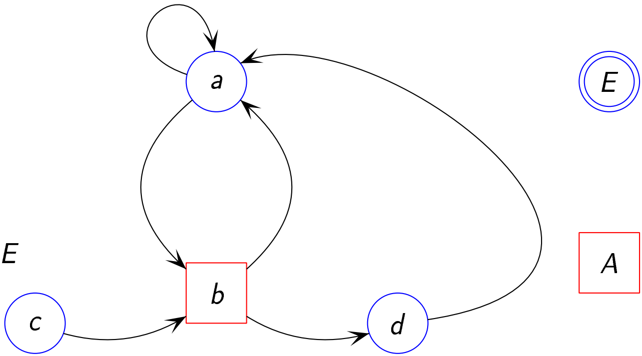
$\{a\} \to c$

$\{a, b\} \to d$

$\{a, b, c\} \to A$

$\{a, b, d\} \to A$

$\{a, b, c, d\} \to E$



Eve wins when the token reaches vertex A.

A technical restriction

Limit transitions of the form $P \longrightarrow q$ where $q \in P$ are forbidden.
This ensures that plays can't be longer than ω^ω.
TODO: dessin

Problems

Reachability game of ordinal length

- a graph with limit transitions,
- two players,
- a state to reach.

The questions are:

- is the game determined?
- if yes, which player has a winning strategy?
- can his strategy be computed?

Note that the length of a play is not known in advance: the game stops when one of the players wins.

Results

Theorem

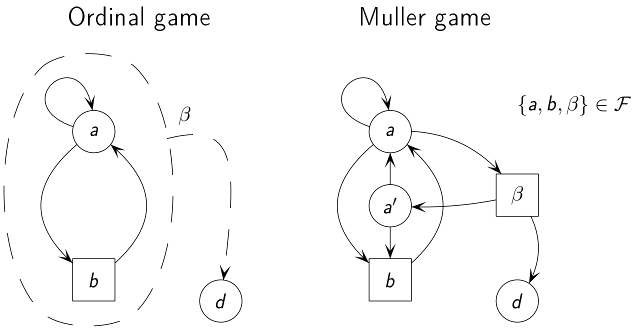
Reachability games of ordinal length $< \omega^\omega$ are determined.

Theorem

Finding the winner is PSPACE*-complete.*

Idea of the proof: the game is reduced to a Muller game, which we know how to solve.

Reduction



There is a translation from a winning strategy in the Muller game to a winning strategy in the ordinal game.

Conclusion

Results:

- One of the players always wins
- Finding the winner with same complexity as for traditional Muller games

Remaining questions:

- How much memory is needed? Are there classes where it is finite?
- Can we lift the restriction to ordinals $< \omega^\omega$?