Graph games on ordinals

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Outline

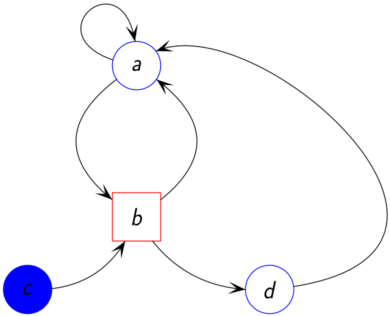
Two player games

- Verification of open systems, controller synthesis
  - One player (Eve) corresponds to the system, the opponent (Adam) represents the hostile environment
  - Winning condition: specification of the system
  - Strategy for Eve: controller ensuring that the spec is met

- Length of plays
  - finite: Interactions limited in depth
  - infinite: Reactive systems
  - ordinal: Timed systems with potential Zeno behaviours

Plays

- Finite graph $G = (V, E)$
- Partition $V = V_E \cup V_A$
- 2 players, Eve and Adam; Eve plays in V_E and Adam in V_A
- Winning condition



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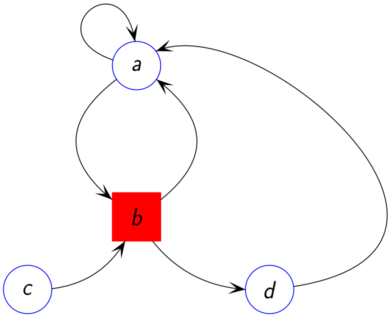

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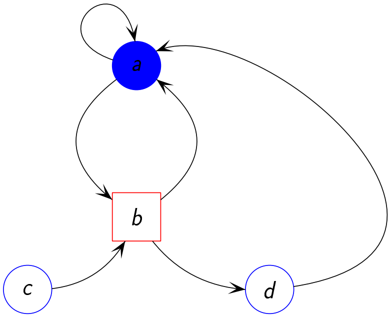

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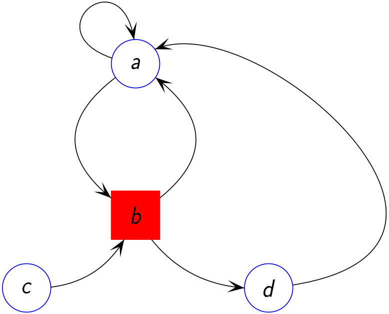

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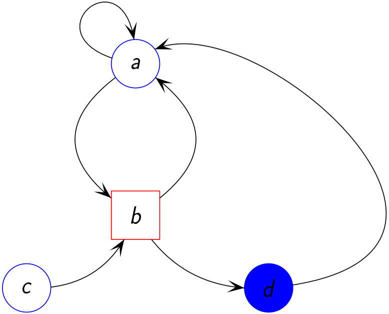

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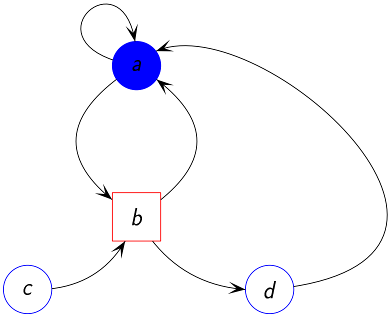


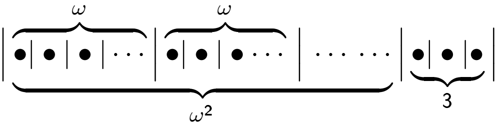
Beyond ω

We want models of systems where infinitely many actions can happen in finite time (Zeno behaviours).

A play is now a word of ordinal length, such as $((ab)^\omega c)^\omega (ba)^\omega d$

Examples:

- ω
- $\omega^2 + 3$



Why ordinals?

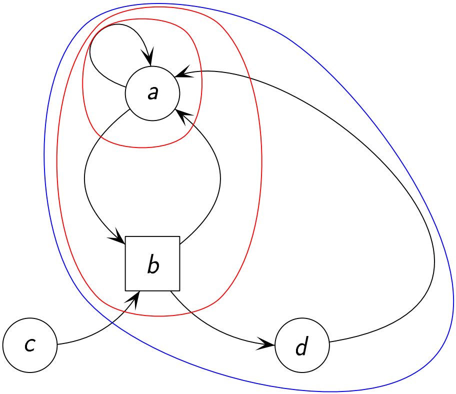
Extension of Church's problem
(Rabinovich & Shomrat)

Automata on ordinal words
(Büchi)

Timed systems
(ordinals allow to consider Zeno behaviours)

Muller games

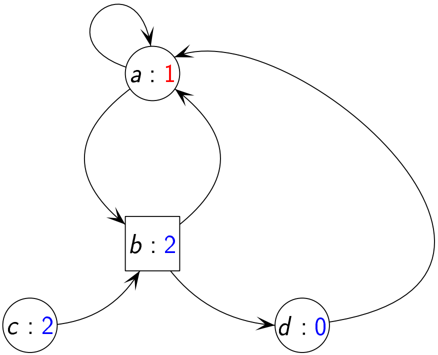
Winning condition: Eve wins if the set of states visited infinitely often is in $\mathcal{F}$.



A play is an infinite word, like $cbabdababdabdabababababda\ldots$.

Parity games

Winning condition: Eve wins if the least colour visited infinitely often is even.



A play is an infinite word, like *cbabdababdabdabababababda* . . . .

Problems

A game is given by a partitioned graph and a winning condition.

We want to know:

- whether the game is determined (one of the players has a winning strategy)
- given an initial state, which is the winning player
- is there a finitely-representable winning strategy
- how to compute such a winning strategy

Theorem

Muller games are determined (Martin).
Finding the winner is PSPACE*-complete (Hunter and Dawar).*

Extending Muller games

We add limit transitions to the arena.

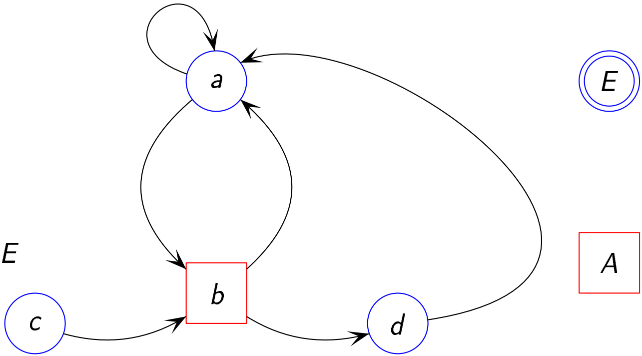
$\{a\} \longrightarrow c$

$\{a, b\} \longrightarrow d$

$\{a, b, c\} \longrightarrow A$

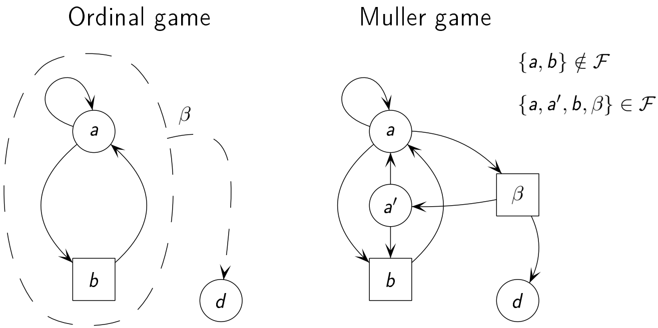
$\{a, b, d\} \longrightarrow A$

$\{a, b, c, d\} \longrightarrow E$



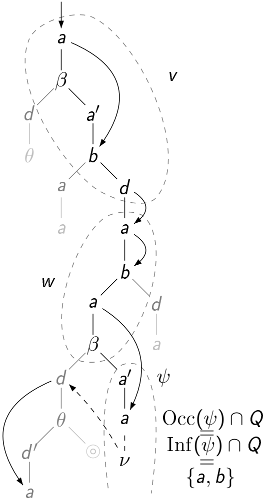
Winning condition: Eve wins when the token reaches vertex E.

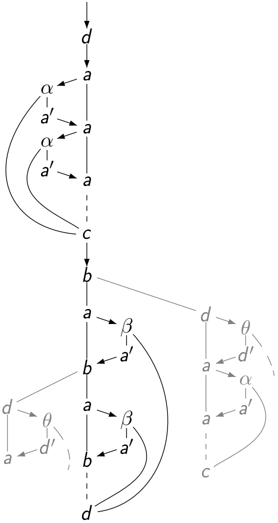
Reduction



A winning strategy in the Muller game corresponds to a winning strategy in the ordinal game.

Translating strategies





Results

Determinacy

Direct consequence of the strategy translation

Finding the winner is PSPACE-complete

Same complexity as traditional Muller games

But..

- Restricted to certain arenas: no limit transitions of the form $P \rightarrow q \in P$
- Strategies need infinite memory

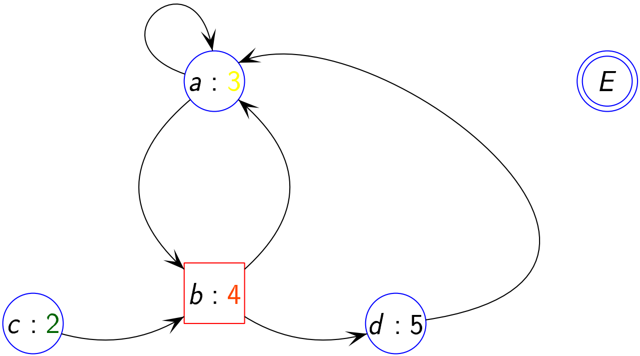
Special case: priority transitions

We add priorities to the states, and limit transitions to the arena.

$2 \longrightarrow E$

$3 \longrightarrow d$

$4 \longrightarrow c$



Winning condition: Eve wins when the token reaches vertex E.

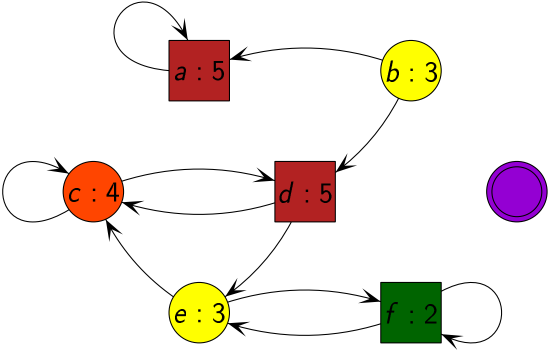
Results

Theorem
Reachability games of ordinal length are determined.

Theorem
In an ordinal priority game, finding the winner is NP $\cap$ co-NP.
The winning player has a positional strategy.

Corollary
In an ordinal Muller game with n vertices, the winner has a strategy with $n!$ memory states.

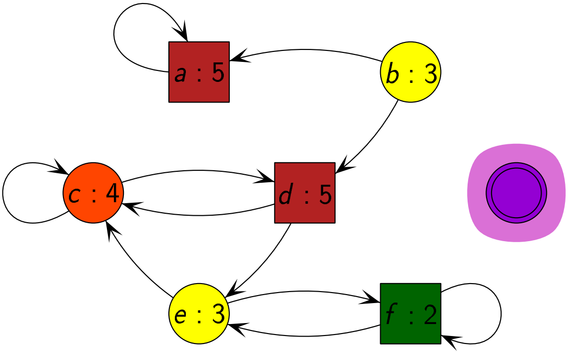
Positional strategies for priority games



$2 \xrightarrow{\lim} a \qquad 3 \xrightarrow{\lim} \odot \qquad 4 \xrightarrow{\lim} d \qquad 5 \xrightarrow{\lim} b$

One can compute winning strategies using a variant of Zielonka's algorithm. These strategies are positional.
Idea: compute successive attractors and refine until we have the winning regions.

Positional strategies for priority games



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0	1	2	3	4	5
E $\odot$	A	A	A	A	A

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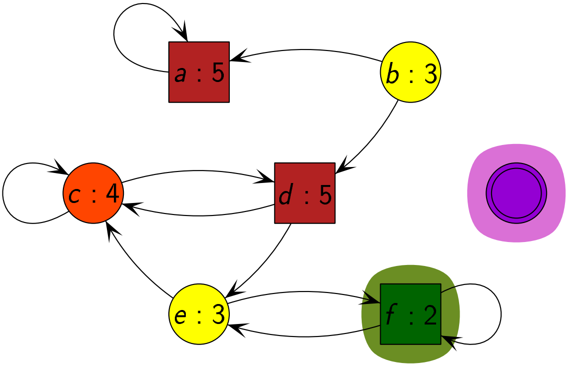

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$\odot$		f			

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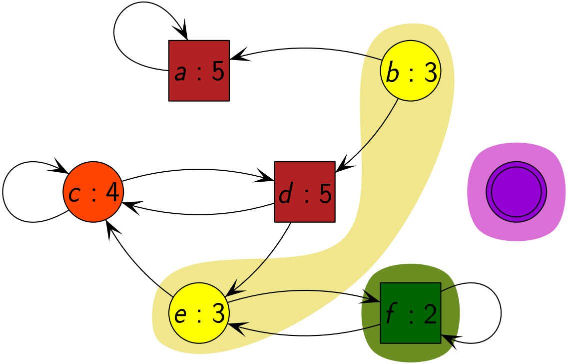

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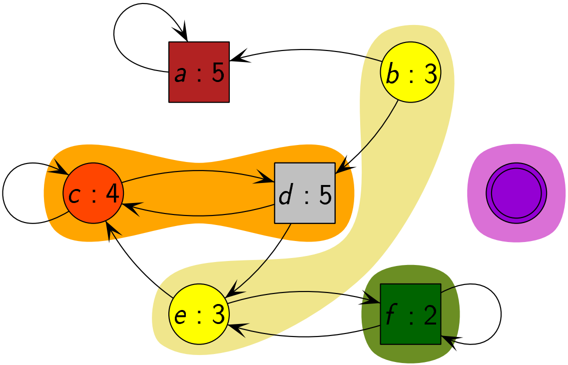

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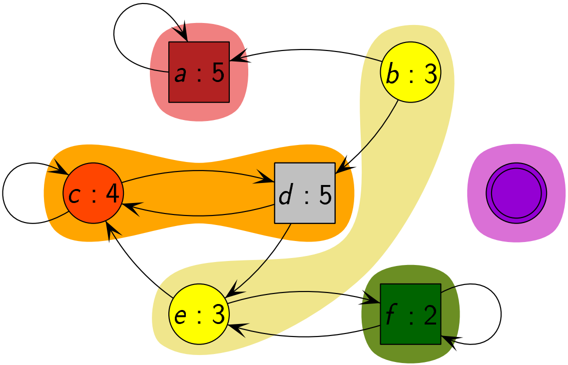

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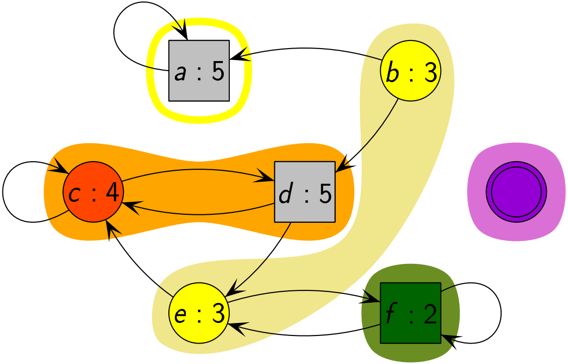

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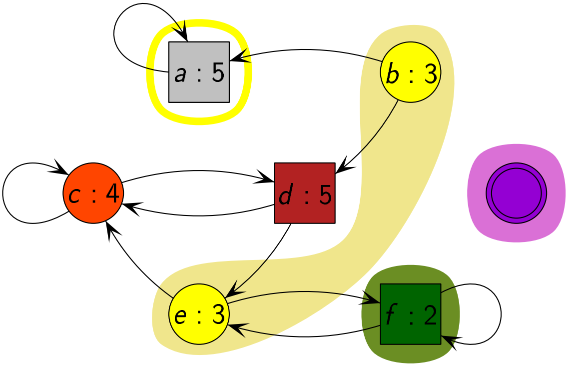

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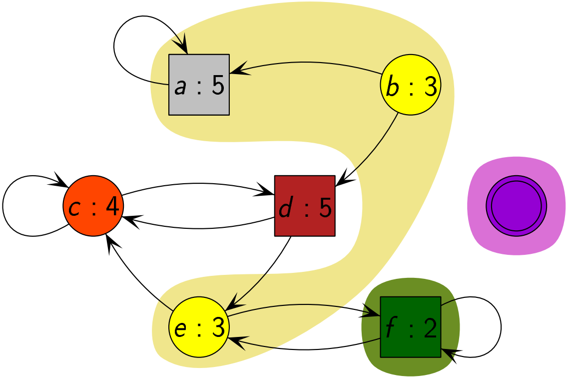

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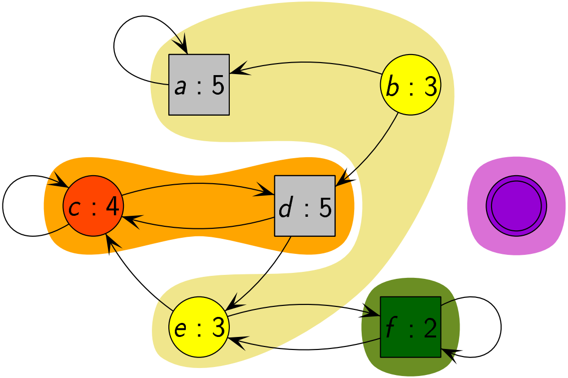

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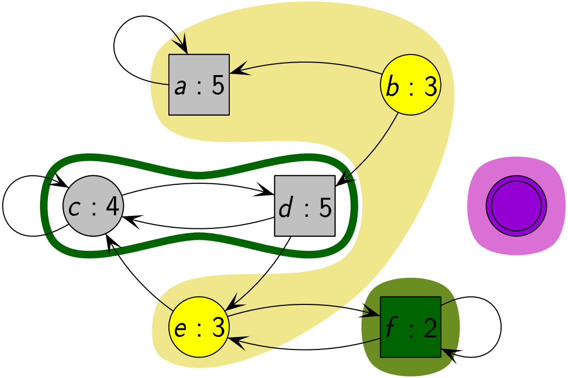

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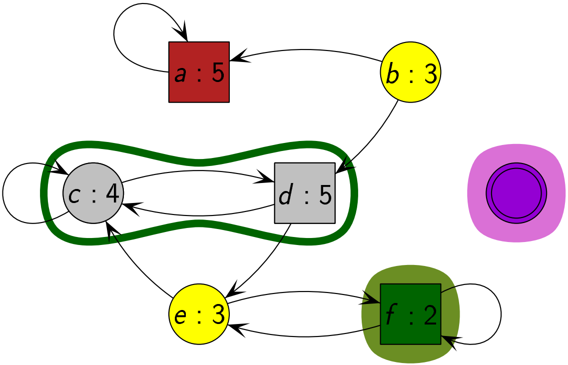

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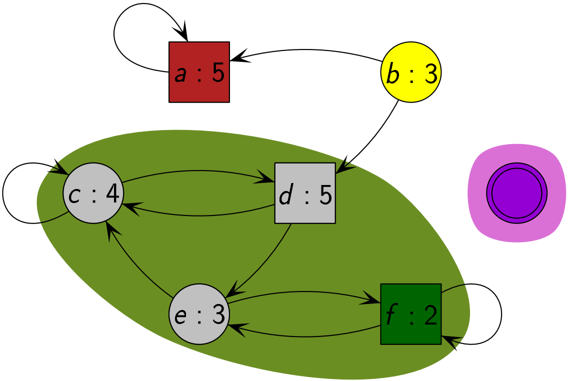

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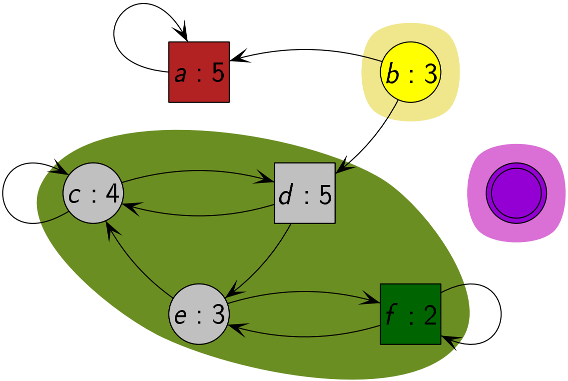

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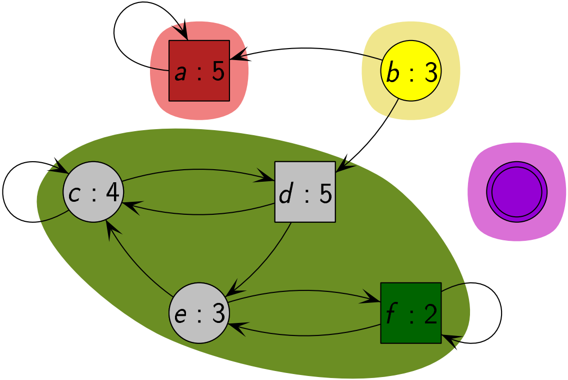

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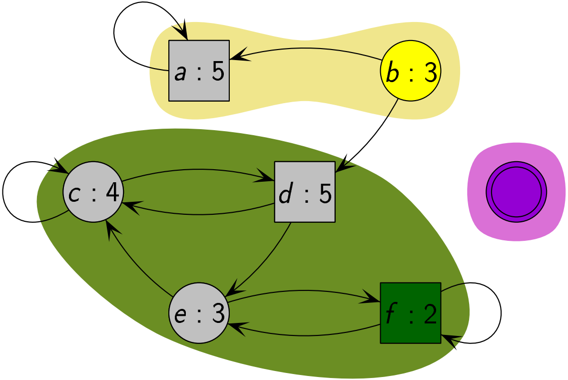

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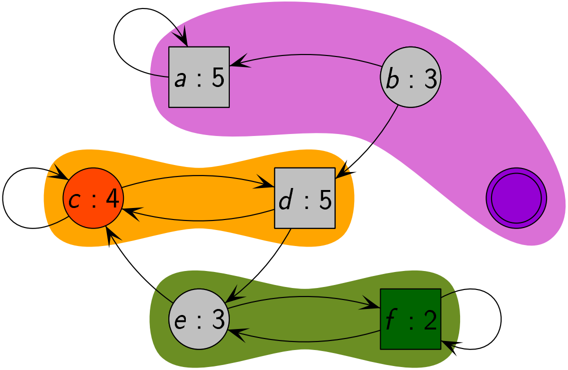
Positional strategies for priority games



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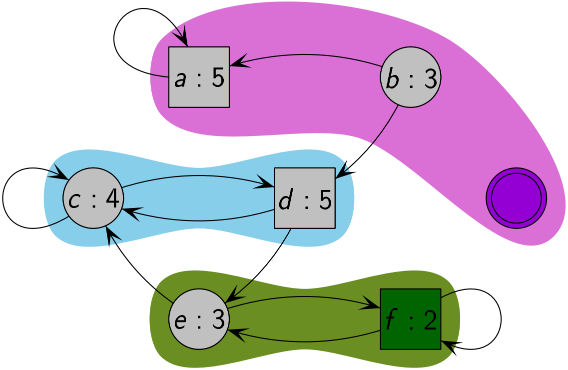

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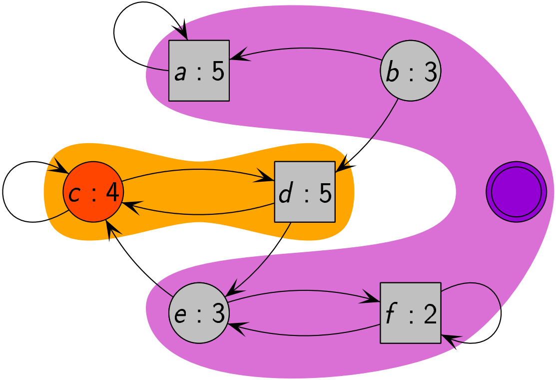

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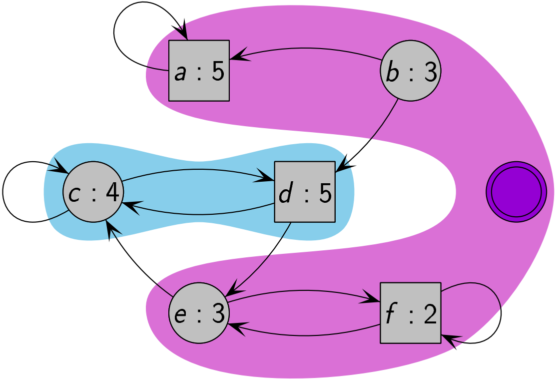

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0	1	2	3	4	5
E $\odot, a, b, e, f$	A c, d	A	A	A	A

The LAR reduction

Latest Appearance Record

A pair (π, i) where:

- π is a permutation over the states
- $1 \leq i \leq \#\text{states}$

States of the reduced (priority) game = LARs of the original game

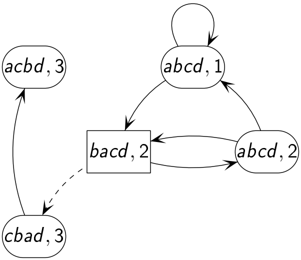
Successor transitions

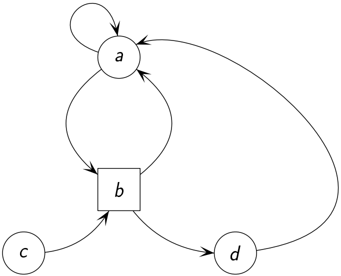
$(\pi, i) \rightarrow (\mu, j)$ if:

- $\pi(1) \rightarrow \mu(1)$
- $\mu(1) = \pi(j)$
- all other states stay in the same order

We need one colour for each LAR.

Detail of the transitions





Conclusion

Determinacy

One of the players has a winning strategy

Complexity

Finding the winner is PSPACE-complete for Muller-like games, and NP $\cap$ co-NP for priority games.

Strategies

Positional strategies in priority games, finite memory in Muller-like games through a reduction